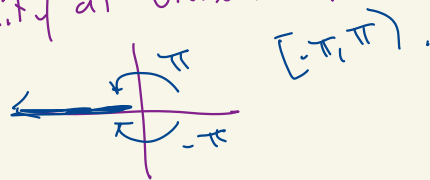


Monday, October 6, Lecture 16: Holomorphic functions,

Last time...

- De Moivre's Thm: $z, re^{i\theta} \in \mathbb{C}$.
 $z^n = re^{i\theta}$ has exactly n solns,
 $z_k = r^{1/n} e^{i(\frac{\theta}{n} + \frac{2\pi k}{n})}, k=0, 1, \dots, n-1$.
- Properties of e .
- Pick a branch $[a, a+2\pi)$. Then
 $\log z = \log|z| + i \arg z$, where $\arg z \in [a, a+2\pi)$.
- Discontinuity at branch cut:



We can now define complex powers.

Def:

choose a branch of $\log$. Then for any $a, b \in \mathbb{C}, a \neq 0$,

$$a^b = e^{b \log a}$$

How does choice of branch affect possible values?

- When a^b has distinct values, they are written as $a^b e^{2\pi n b i}$, $n \in \mathbb{Z}$ } ← Most important part!

3 options:

By considering different branches of $\log$...

i) a^b is independent of choice of branch iff $b \in \mathbb{Z}$.

This is just stuff like a^3 or a^{-2} .

ii) a^b takes on exactly q different values iff $b = \frac{p}{q}$ (with p, q having no common factors)

Think: De Moivre

$z^n = w$ has n solutions,

so $w^{1/n}$ has n possible values,

Choice of branch chooses them,

Ex: $z^2 = -1$ has 2 solutions, i and $-i$.

iii) a^b takes on infinitely many values iff b is irrational, or $\text{Im}(b) \neq 0$.

This one is fundamentally new.

- Hw: calculate i^i using multiple branches and compare.

- Ex: Consider i^θ where θ is irrational. Then possible values can be written in terms of the principal branch $(-\pi, \pi)$ of $\log$:
$$e^{\theta(\log i + 2\pi n i)}, n \in \mathbb{Z}.$$

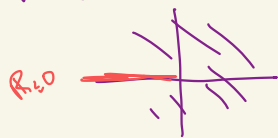
Since θ irrational, each $\theta \cdot 2\pi n \neq \theta \cdot 2\pi m \pmod{2\pi}$. All distinct!

pf:

Choose some branch, e.g. $[-\pi, \pi)$, and let $\log z$ correspond to this branch.

Any other branch is $\log a + 2\pi ni$, $n \in \mathbb{Z}$.

[Note: we can think of this as choosing a cut, like $R_{\leq 0}$.



This cut forces the branches to be labelled by integers, $[-\pi + 2\pi n, \pi + 2\pi n)$

..., $[-5\pi, -3\pi)$, $[-3\pi, -\pi)$, $[-\pi, \pi)$, $[\pi, 3\pi)$, ...
 $n = -2$ $n = -1$ $n = 0$ $n = 1$

L

(principal branch, $\log$)

$$a^b = e^{b(\log a + 2\pi ni)} = e^{b \log a} e^{2\pi n b i}$$

i) if $b \in \mathbb{Z}$, then $nb \in \mathbb{Z}$ so $e^{2\pi i nb} = 1$,

ii) if $b = \frac{p}{q}$, then $e^{2\pi i n p/q}$ has q distinct values,
 $n = 0, 1, \dots, q-1$.

iii) if b irrational, then each $e^{2\pi i nb}$ must be distinct:
towards contradiction, if $\exists n \neq m$ with

$$e^{2\pi i nb} = e^{2\pi i mb} \Leftrightarrow e^{2\pi i b(n-m)} = 1$$

but b is irrational, so it cannot be that $b(n-m) \in \mathbb{Z}$.

• if $\text{Im}(b) \neq 0$, then $b = x + iy$, so

$$\begin{aligned} |e^{2\pi i n(x+iy)}| &= |e^{2\pi i nx} e^{-2\pi ny}| \\ &= e^{-2\pi ny} \end{aligned}$$

which evidently takes a different value for each $n \in \mathbb{Z}$.



Def:

Let $f: \Omega \rightarrow \mathbb{C}$

The derivative of f at z_0 is the limit

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \left(\text{also, } \frac{df}{dz}(z_0) \right).$$

- We call a function holomorphic at z_0 if $f'(z_0)$ exists.

Thm (deep fact)

f is holomorphic on a small disc near z_0

$$\Leftrightarrow f \text{ is analytic at } z_0, \text{ i.e. } f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ with some nonzero radius of convergence } R.$$

[Warning ^(Taylor series): There are plenty of functions which are smooth but not analytic, and they can be physically relevant!]

Many of the usual suspects are the same.

$$\frac{d}{dz} e^z = e^z, \quad \frac{d}{dz} \sin z = \cos z, \quad \frac{d}{dz} z^n = n z^{n-1} \text{ for } n \geq 1 \dots$$

Ex: $f(z) = \frac{1}{z}$, $f: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$.

pick $z_0 \neq 0$.

$$\begin{aligned} f'(z_0) &= \lim_{h \rightarrow 0} \frac{f(z_0+h) - f(z_0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{z_0+h} - \frac{1}{z_0}}{h} \\ &= \lim_{h \rightarrow 0} \frac{z_0 - (z_0+h)}{h(z_0+h)z_0} \\ &= \lim_{h \rightarrow 0} \frac{-1}{(z_0+h)z_0} \\ &= -\frac{1}{z_0^2} \end{aligned}$$

• We can always think of

$$f: \mathbb{C} \rightarrow \mathbb{C}$$

$$\text{as } f: \mathbb{R}^2 \rightarrow \mathbb{R}^2.$$

Fact: every holomorphic f is real-differentiable,
but not vice versa.

Holomorphic = Real differentiable + "Extra structure"

Ex:

$f: \mathbb{C} \rightarrow \mathbb{C}$ given by $f(z) = |z|^2$ is not holomorphic at any $z \neq 0$.

It is real differentiable at $(x, y) \neq 0$:

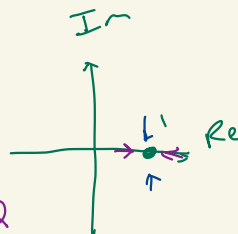
$$f(x, y) = \sqrt{x^2 + y^2}$$

Observe: Let's check at $z=1$.

Note that if we take the limit along $h \in \mathbb{R}$,

$$\lim_{h \rightarrow 0} \frac{|z+h|^2 - |z|^2}{h} = \lim_{h \rightarrow 0} \frac{z\bar{z} + h(z+\bar{z}) + h^2 - z\bar{z}}{h} = z + \bar{z} = 2$$

$\uparrow$
 $z=1$



But along $ih \in i\mathbb{R}$

$$\lim_{ih \rightarrow 0} \frac{|z+ih|^2 - |z|^2}{ih} = \lim_{ih \rightarrow 0} \frac{(z+ih)(\bar{z}-ih) - z\bar{z}}{ih} = \lim_{ih \rightarrow 0} \frac{ih(\bar{z}-z) - h^2}{ih}$$

$z=1 \rightarrow = 0$

$\Rightarrow$ Limit does not agree in all directions, so DNE.

Complex derivatives are special: limits can now "move" in many directions!